

# On the well-posedness of the Ideal MHD equations in the Triebel-Lizorkin spaces

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## Abstract

In this paper, we prove the local well-posedness for the Ideal MHD equations in the Triebel-Lizorkin spaces and obtain blow-up criterion of smooth solutions. Specially, we fill a gap in a step of the proof of the local well-posedness part for the incompressible Euler equation in [7].

**Key words.** Ideal MHD equations, well-posedness, bow-up criterion, particle trajectory mapping, para-differential calculus, Triebel-Lizorkin space

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## 1 Introduction

In this paper, we are concerned with the Ideal MHD equations in  $\mathbf{R}^d$ :

$$(IMHD) \quad \begin{cases} u_t + u \cdot \nabla u = -\nabla p - \frac{1}{2} \nabla b^2 + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(0, x) = u_0(x), \quad b(0, x) = b_0(x), \end{cases} \quad (1.1)$$

where  $x \in \mathbf{R}^d, t \geq 0$ ,  $u, b$  describes the flow velocity vector and the magnetic field vector respectively,  $p$  is a scalar pressure, while  $u_0$  and  $b_0$  are the given initial velocity and initial magnetic field with  $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ .

Using the standard energy method [14], it can be proved that for  $(u_0, b_0) \in H^s(\mathbf{R}^d)$ ,  $s > \frac{d}{2} + 1$ , there exists  $T > 0$  such that the Cauchy problem (1.1) has a unique smooth solution  $(u(t, x), b(t, x))$  on  $[0, T)$  satisfying

$$(u, b) \in C([0, T); H^s) \cap C^1([0, T); H^{s-1}).$$

But whether this local solution will exist globally or lead to a singularity in finite time is still an outstanding open problem. Caffisch, Klapper and Steele[4] extended Beale-Kato-Majda criterion [2] for the incompressible Euler equations to the Ideal MHD equations. More

precisely, they showed if the smooth solution  $(u, b)$  satisfies the following condition:

$$\int_0^T (\|\nabla \times u\|_{L^\infty} + \|\nabla \times b\|_{L^\infty}) dt < \infty, \quad (1.2)$$

then the solution  $(u, b)$  can be extended beyond  $t = T$ , namely, for some  $T < \tilde{T}$ ,  $(u, b) \in C([0, \tilde{T}); H^s) \cap C^1([0, \tilde{T}); H^{s-1})$ . One can refer to [5, 23] for the other refined criterions, and for the viscous MHD equations, some criterions can be found in [6, 19, 20, 21, 22].

Recently, Chae studied the local well-posedness and blow-up criterion for the incompressible Euler equations in the Triebel-Lizorkin spaces [7, 8]. As we know, Triebel-Lizorkin spaces are the unification of several classical function spaces such as Lebesgue spaces  $L^p(\mathbf{R}^d)$ , Sobolev spaces  $H_p^s(\mathbf{R}^d)$ , Lipschitz spaces  $C^s(\mathbf{R}^d)$ , and so on. In [7], the author first used the Littlewood-Paley operator to localize the Euler equation to the frequency annulus  $\{|\xi| \sim 2^j\}$ , then obtained an integral representation of the frequency-localized solution on the Lagrangian coordinates by introducing a family of particle trajectory mapping  $\{X_j(\alpha, t)\}$  defined by

$$\begin{cases} \frac{\partial}{\partial t} X_j(\alpha, t) = (S_{j-2}v)(X_j(\alpha, t), t) \\ X_j(\alpha, 0) = \alpha, \end{cases} \quad (1.3)$$

where  $v$  is a divergence-free velocity field and  $S_{j-2}$  is a frequency projection to the ball  $\{|\xi| \lesssim 2^j\}$  (see Section 2).

With the integral representation, one can obtain the well-posedness of the Euler equation in the framework of the Besov spaces by standard argument, due to the following important relation

$$\left( \sum_{j \in \mathbf{Z}} 2^{jsq} \|\Delta_j v(X_j(\alpha, t))\|_{L^p(\cdot d\alpha)}^q \right)^{\frac{1}{q}} \cong \|v\|_{\dot{B}_{p,q}^s}$$

by the volume-preserving property of the mapping  $\{X_j(\alpha, t)\}$  which is defined by (1.3). However, if we work in the framework of the Triebel-Lizorkin spaces, and the trajectory mapping  $\{X_j(\alpha, t)\}$  is taken, we don't know whether the relation

$$\left\| \left( \sum_{j \in \mathbf{Z}} 2^{jsq} |\Delta_j v(X_j(\alpha, t))|^q \right)^{\frac{1}{q}} \right\|_{L^p(\cdot d\alpha)} \cong \left\| \left( \sum_{j \in \mathbf{Z}} 2^{jsq} |\Delta_j v(x)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\cdot dx)} = \|v\|_{\dot{F}_{p,q}^s} \quad (1.4)$$

holds. The reason is that the mapping  $\{X_j(\alpha, t)\}$  depends on the index  $j$ , and we can't find a uniform change of the coordinates independent of  $j$  such that (1.4) holds. On the other hand, the proof of the commutator estimate (the key point of the proof of the local well-posedness part)

$$\left\| \left( \sum_{j \in \mathbf{Z}} 2^{jsq} |[(S_{j-2}v \cdot \nabla) \Delta_j v - \Delta_j((v \cdot \nabla)v)](X_j(\alpha, t))|^q \right)^{\frac{1}{q}} \right\|_{L^p} \leq C \|\nabla v\|_\infty \|v\|_{\dot{F}_{p,q}^s} \quad (1.5)$$

also leads to some trouble due to similar reasons.

The purpose of this paper is to deal with the well-posedness of the Ideal MHD equations (1.1) in the Triebel-Lizorkin spaces. Firstly, we can reduce (1.1) to the transport equations by introducing the symmetrizers. If we still use the trajectory mapping depending on  $j$ , the

above-mentioned trouble will occur. In order to overcome this difficulty, we will introduce a different family of particle trajectory mapping  $\{X(\alpha, t)\}$  independent of  $j$  defined by

$$\begin{cases} \frac{\partial}{\partial t} X(\alpha, t) = v(X(\alpha, t), t) \\ X(\alpha, 0) = \alpha. \end{cases}$$

The price to pay here is that we have to establish the following commutator estimate

$$\left\| \left( \sum_{j \in \mathbf{Z}} 2^{jsq} |[(v \cdot \nabla) \Delta_j u - \Delta_j((v \cdot \nabla)u)]|^q \right)^{\frac{1}{q}} \right\|_{L^p} \leq C (\|\nabla v\|_{L^\infty} \|u\|_{\dot{F}_{p,q}^s} + \|u\|_{L^\infty} \|\nabla v\|_{\dot{F}_{p,q}^s})$$

by the paradifferential calculus, whose proof is more complicated since  $v$  is rougher than  $S_{j-2}v$  which is the smooth low frequency cut-off of  $v$ . It is necessary to point out that the Maximal inequality (see Lemma 2.5 in Section 2) plays a key role in the proof of the above inequality, which helps us to avoid other difficulties arising from the change of the coordinates.

Now we state our result as follows.

**Theorem 1.1 (i) Local-in-time Existence.** *Let  $(u_0, b_0) \in F_{p,q}^s$ ,  $s > \frac{d}{p} + 1$ ,  $1 < p, q < \infty$  satisfying  $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ . Then there exists  $T = T(\|(u_0, b_0)\|_{F_{p,q}^s})$  such that the IMHD has a unique solution  $(u, b) \in C([0, T]; F_{p,q}^s)$ .*

**(ii) Blow-up Criterion.** *The local-in-time solution  $(u, b) \in C([0, T]; F_{p,q}^s)$  constructed in (i) blows up at  $T^* > T$  in  $F_{p,q}^s$ , i.e.*

$$\limsup_{t \nearrow T^*} \|(u, b)\|_{F_{p,q}^s} = +\infty, \quad T^* < \infty,$$

if and only if

$$\int_0^{T^*} \|(\nabla \times u, \nabla \times b)(t)\|_{\dot{F}_{\infty,\infty}^0} dt = +\infty. \quad (1.6)$$

**Remark 1.2** *In the case of  $b = 0$ , (IMHD) can be read as the incompressible Euler equations, and what proved in [7] is a straightforward consequence of Theorem 1.1.*

**Remark 1.3** *Using the argument in [5], we can also refine blow-up criterion (1.6) to the following form: there exists a positive constant  $M_0$  such that if*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{j \in \mathbf{Z}} \int_{T^* - \varepsilon}^{T^*} \|(\Delta_j(\nabla \times u), \Delta_j(\nabla \times b))(t)\|_{L^\infty} dt \geq M_0,$$

*then the smooth solution  $(u, b)$  blows up at  $t = T^*$ .*

**Notation:** Throughout this paper,  $C$  stands for a “harmless” constant, and we will use the notation  $A \lesssim B$  as an equivalent to  $A \leq CB$ ,  $A \approx B$  as  $A \lesssim B$  and  $B \lesssim A$ , and denote  $\|\cdot\|_p$  by  $L^p(\mathbf{R}^d)$  norm of a function.

## 2 Preliminaries

Let  $\mathcal{B} = \{\xi \in \mathbf{R}^d, |\xi| \leq \frac{4}{3}\}$  and  $\mathcal{C} = \{\xi \in \mathbf{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ . Choose two nonnegative smooth radial functions  $\chi, \varphi$  supported respectively in  $\mathcal{B}$  and  $\mathcal{C}$  such that

$$\begin{aligned}\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbf{R}^d, \\ \sum_{j \in \mathbf{Z}} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbf{R}^d \setminus \{0\}.\end{aligned}$$

We denote  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ ,  $h = \mathcal{F}^{-1}\varphi$  and  $\tilde{h} = \mathcal{F}^{-1}\chi$ . Then the dyadic blocks  $\Delta_j$  and  $S_j$  can be defined as follows

$$\begin{aligned}\Delta_j f &= \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbf{R}^d} h(2^j y) f(x-y) dy, \\ S_j f &= \sum_{k \leq j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{jd} \int_{\mathbf{R}^d} \tilde{h}(2^j y) f(x-y) dy.\end{aligned}$$

Formally,  $\Delta_j = S_j - S_{j-1}$  is a frequency projection to the annulus  $\{|\xi| \sim 2^j\}$ , and  $S_j$  is a frequency projection to the ball  $\{|\xi| \lesssim 2^j\}$ . One easily verifies that with our choice of  $\varphi$

$$\Delta_j \Delta_k f \equiv 0 \quad \text{if} \quad |j-k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) \equiv 0 \quad \text{if} \quad |j-k| \geq 5. \quad (2.1)$$

With the introduction of  $\Delta_j$  and  $S_j$ , let us recall the definition of the Triebel-Lizorkin space. Let  $s \in \mathbf{R}$ ,  $(p, q) \in [1, \infty) \times [1, \infty]$ , the homogenous Triebel-Lizorkin space  $\dot{F}_{p,q}^s$  is defined by

$$\dot{F}_{p,q}^s = \{f \in \mathcal{Z}'(\mathbf{R}^d); \|f\|_{\dot{F}_{p,q}^s} < \infty\},$$

where

$$\|f\|_{\dot{F}_{p,q}^s} = \begin{cases} \left\| \left( \sum_{j \in \mathbf{Z}} 2^{jsq} |\Delta_j f|^q \right)^{\frac{1}{q}} \right\|_p, & \text{for } 1 \leq q < \infty, \\ \left\| \sup_{j \in \mathbf{Z}} (2^{js} |\Delta_j f|) \right\|_p, & \text{for } q = \infty, \end{cases}$$

and  $\mathcal{Z}'(\mathbf{R}^d)$  denotes the dual space of  $\mathcal{Z}(\mathbf{R}^d) = \{f \in \mathcal{S}(\mathbf{R}^d); \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbf{N}^d \text{ multi-index}\}$  and can be identified by the quotient space of  $\mathcal{S}'/\mathcal{P}$  with the polynomials space  $\mathcal{P}$ .

For  $s > 0$ , and  $(p, q) \in [1, \infty) \times [1, \infty]$ , we define the inhomogeneous Triebel-Lizorkin space  $F_{p,q}^s$  as follows

$$F_{p,q}^s = \{f \in \mathcal{S}'(\mathbf{R}^d); \|f\|_{F_{p,q}^s} < \infty\},$$

where

$$\|f\|_{F_{p,q}^s} = \|f\|_p + \|f\|_{\dot{F}_{p,q}^s}.$$

We refer to [1, 18] for more details.

**Lemma 2.1** (*Bernstein's inequality*) *Let  $k \in \mathbf{N}$ . There exist a constant  $C$  independent of  $f$  and  $j$  such that for all  $1 \leq p \leq q \leq \infty$ , the following inequalities hold:*

$$\begin{aligned}\text{supp } \hat{f} \subset \{|\xi| \lesssim 2^j\} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_q \leq C 2^{jk+jd(\frac{1}{p}-\frac{1}{q})} \|f\|_p, \\ \text{supp } \hat{f} \subset \{|\xi| \sim 2^j\} &\Rightarrow \|f\|_p \leq C \sup_{|\alpha|=k} 2^{-jk} \|\partial^\alpha f\|_p.\end{aligned}$$

For the proof, see [10, 15].

**Lemma 2.2** *For any  $k \in \mathbf{N}$ , there exists a constant  $C_k$  such that the following inequality holds:*

$$C_k^{-1} \|\nabla^k f\|_{\dot{F}_{p,q}^s} \leq \|f\|_{\dot{F}_{p,q}^{s+k}} \leq C_k \|\nabla^k f\|_{\dot{F}_{p,q}^s}.$$

The proof can be found in [18].

**Proposition 2.3** [7] *Let  $s > 0$ ,  $(p, q) \in (1, \infty) \times (1, \infty]$ , or  $p = q = \infty$ , then there exists a constant  $C$  such that*

$$\begin{aligned} \|fg\|_{\dot{F}_{p,q}^s} &\leq C(\|f\|_\infty \|g\|_{\dot{F}_{p,q}^s} + \|g\|_\infty \|f\|_{\dot{F}_{p,q}^s}), \\ \|fg\|_{F_{p,q}^s} &\leq C(\|f\|_\infty \|g\|_{F_{p,q}^s} + \|g\|_\infty \|f\|_{F_{p,q}^s}). \end{aligned}$$

For a locally integrable function  $f$ , the maximal function  $Mf(x)$  is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|\mathcal{B}(x, r)|} \int_{\mathcal{B}(x, r)} |f(y)| dy,$$

where  $|\mathcal{B}(x, r)|$  is the volume of the ball  $\mathcal{B}(x, r)$  with center  $x$  and radius  $r$ .

**Lemma 2.4** [11] *(Vector Maximal inequality) Let  $(p, q) \in (1, \infty) \times (1, \infty]$  or  $p = q = \infty$  be given. Suppose  $\{f_j\}_{j \in \mathbf{Z}}$  is a sequence of function in  $L^p$  with the property that  $\|f_j(x)\|_{\ell^q(\mathbf{Z})} \in L^p(\mathbf{R}^d)$ . Then there holds*

$$\left\| \left( \sum_{j \in \mathbf{Z}} |Mf_j(x)|^q \right)^{\frac{1}{q}} \right\|_p \leq C \left\| \left( \sum_{j \in \mathbf{Z}} |f_j(x)|^q \right)^{\frac{1}{q}} \right\|_p.$$

**Lemma 2.5** *Let  $\varphi$  be an integrable function on  $\mathbf{R}^d$ , and set  $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi(\frac{x}{\varepsilon})$  for  $\varepsilon > 0$ . Suppose that the least decreasing radial majorant of  $\varphi$  is integrable; i.e. let*

$$\psi(x) = \sup_{|y| \geq |x|} |\varphi(y)|,$$

*and we suppose  $\int_{\mathbf{R}^d} \psi(x) dx = A < \infty$ . Then with the same  $A$ , for  $f \in L^p(\mathbf{R}^d)$ ,  $1 \leq p \leq \infty$*

$$\sup_{\varepsilon > 0} |(f * \varphi_\varepsilon)(x)| \leq AM(f)(x).$$

The proof can be found in [17], Chap. III.

### 3 The proof of Theorem 1.1

We divide the proof of Theorem 1.1 into several steps.

**Step 1.** A priori estimates.

Let us symmetrize the equation (1.1). Set

$$z^+ = u + b, \quad z^- = u - b,$$

then (1.1) can be reduced to the system for  $z^+$  and  $z^-$

$$\begin{cases} \partial_t z^+ + (z^- \cdot \nabla) z^+ = -\nabla \pi, \\ \partial_t z^- + (z^+ \cdot \nabla) z^- = -\nabla \pi, \\ \nabla \cdot z^+ = \nabla \cdot z^- = 0, \\ z^+(0) = z_0^+ = u_0 + b_0, \quad z^-(0) = z_0^- = u_0 - b_0, \end{cases} \quad (3.1)$$

where  $\pi = p + \frac{1}{2}b^2$ . Taking the operation  $\Delta_k$  on both sides of (3.1), we get

$$\begin{cases} \partial_t \Delta_k z^+ + z^- \cdot \nabla \Delta_k z^+ + \nabla \Delta_k \pi = [z^-, \Delta_k] \cdot \nabla z^+, \\ \partial_t \Delta_k z^- + z^+ \cdot \nabla \Delta_k z^- + \nabla \Delta_k \pi = [z^+, \Delta_k] \cdot \nabla z^-, \end{cases} \quad (3.2)$$

where we denote the commutators

$$\begin{aligned} [z^-, \Delta_k] \cdot \nabla z^+ &\triangleq z^- \cdot \nabla \Delta_k z^+ - \Delta_k((z^- \cdot \nabla) z^+), \\ [z^+, \Delta_k] \cdot \nabla z^- &\triangleq z^+ \cdot \nabla \Delta_k z^- - \Delta_k((z^+ \cdot \nabla) z^-). \end{aligned}$$

Let  $X_t^+(\alpha)$  and  $X_t^-(\alpha)$  be the solutions of the following ordinary differential equations:

$$\begin{cases} \partial_t X_t^+(\alpha) = z^-(X_t^+(\alpha), t), \\ \partial_t X_t^-(\alpha) = z^+(X_t^-(\alpha), t), \\ X_t^+(\alpha)|_{t=0} = X_t^-(\alpha)|_{t=0} = \alpha. \end{cases} \quad (3.3)$$

Then, it follows from (3.2) that

$$\begin{cases} \frac{d}{dt} \Delta_k z^+(X_t^+(\alpha), t) = [z^-, \Delta_k] \cdot \nabla z^+(X_t^+(\alpha), t) - \nabla \Delta_k \pi(X_t^+(\alpha), t), \\ \frac{d}{dt} \Delta_k z^-(X_t^-(\alpha), t) = [z^+, \Delta_k] \cdot \nabla z^-(X_t^-(\alpha), t) - \nabla \Delta_k \pi(X_t^-(\alpha), t), \end{cases} \quad (3.4)$$

which implies that

$$\begin{aligned} |\Delta_k z^+(X_t^+(\alpha), t)| &\leq |\Delta_k z_0^+(\alpha)| + \int_0^t |([z^-, \Delta_k] \cdot \nabla z^+)(X_\tau^+(\alpha), \tau)| d\tau \\ &\quad + \int_0^t |\Delta_k \nabla \pi(X_\tau^+(\alpha), \tau)| d\tau. \end{aligned} \quad (3.5)$$

Multiplying  $2^{ks}$ , taking  $\ell^q(\mathbf{Z})$  norm on both sides of (3.5), we get by using Minkowski inequality that

$$\begin{aligned} \left( \sum_k |2^{ks} \Delta_k z^+(X_t^+(\alpha), t)|^q \right)^{\frac{1}{q}} &\leq \left( \sum_k |2^{ks} \Delta_k z_0^+(\alpha)|^q \right)^{\frac{1}{q}} + \int_0^t \left( \sum_k |2^{ks} \Delta_k \nabla \pi(X_\tau^+(\alpha), \tau)|^q \right)^{\frac{1}{q}} d\tau \\ &\quad + \int_0^t \left( \sum_k |2^{ks} ([z^-, \Delta_k] \cdot \nabla z^+)(X_\tau^+(\alpha), \tau)|^q \right)^{\frac{1}{q}} d\tau. \end{aligned} \quad (3.6)$$

Next, taking the  $L^p$  norm with respect to  $\alpha \in \mathbf{R}^d$  on both sides of (3.6), we get by using the Minkowski inequality that

$$\begin{aligned} & \left( \int_{\mathbf{R}^d} \left| \left( \sum_k |2^{ks} \Delta_k z^+(X_t^+(\alpha), t)|^q \right)^{\frac{1}{q}} \right|^p d\alpha \right)^{\frac{1}{p}} \\ & \leq \|z_0^+\|_{\dot{F}_{p,q}^s} + \int_0^t \left( \int_{\mathbf{R}^d} \left| \left( \sum_k |2^{ks} \Delta_k \nabla \pi(X_\tau^+(\alpha), \tau)|^q \right)^{\frac{1}{q}} \right|^p d\alpha \right)^{\frac{1}{p}} d\tau \\ & \quad + \int_0^t \left( \int_{\mathbf{R}^d} \left| \left( \sum_k |2^{ks} ([z^-, \Delta_k] \cdot \nabla z^+)(X_\tau^+(\alpha), \tau)|^q \right)^{\frac{1}{q}} \right|^p d\alpha \right)^{\frac{1}{p}} d\tau. \end{aligned} \quad (3.7)$$

Using the fact that  $X_t^+(\alpha)$  is a volume-preserving diffeomorphism due to  $\operatorname{div} z^+ = 0$ , we get from (3.7) that

$$\begin{aligned} \|z^+(t)\|_{\dot{F}_{p,q}^s} & \leq \|z_0^+\|_{\dot{F}_{p,q}^s} + \int_0^t \|\nabla \pi\|_{\dot{F}_{p,q}^s} d\tau \\ & \quad + \int_0^t \left\| \left\| 2^{ks} ([z^-, \Delta_k] \cdot \nabla z^+) \right\|_{\ell^q(k \in \mathbf{Z})} \right\|_p d\tau. \end{aligned} \quad (3.8)$$

Thanks to Proposition 4.1, the last term on the right side of (3.8) is dominated by

$$\int_0^t (\|\nabla z^+\|_\infty + \|\nabla z^-\|_\infty) (\|z^-\|_{\dot{F}_{p,q}^s} + \|z^+\|_{\dot{F}_{p,q}^s}) d\tau. \quad (3.9)$$

Next, we estimate the second term on the right side of (3.8). Taking the divergence on both sides of (3.1), we obtain the following representation of the pressure

$$\pi = (-\Delta)^{-1} (\partial_j z_i^- \partial_i z_j^+) = (-\Delta)^{-1} \partial_i \partial_j (z_i^- z_j^+). \quad (3.10)$$

For  $l, m \in [1, d]$ , we have

$$\partial_l \partial_m \pi = (-\Delta)^{-1} \partial_l \partial_m (\partial_j z_i^- \partial_i z_j^+) = \mathcal{R}_l \mathcal{R}_m (\partial_j z_i^- \partial_i z_j^+),$$

where  $\mathcal{R}_l$  denotes the Riesz transform. Thanks to the boundedness of the Riesz transform in the homogeneous Triebel-Lizorkin spaces [12], Lemma 2.2 and Proposition 2.3, we get

$$\begin{aligned} \|\nabla \pi\|_{\dot{F}_{p,q}^s} & \leq C \sum_{l,m=1}^d \|\partial_l \partial_m \pi\|_{\dot{F}_{p,q}^{s-1}} \leq C \|\partial_j z_i^- \partial_i z_j^+\|_{\dot{F}_{p,q}^{s-1}} \\ & \leq C (\|\nabla z^-\|_\infty \|\nabla z^+\|_{\dot{F}_{p,q}^{s-1}} + \|\nabla z^+\|_\infty \|\nabla z^-\|_{\dot{F}_{p,q}^{s-1}}) \\ & \leq C (\|\nabla z^-\|_\infty \|z^+\|_{\dot{F}_{p,q}^s} + \|\nabla z^+\|_\infty \|z^-\|_{\dot{F}_{p,q}^s}). \end{aligned} \quad (3.11)$$

Plugging (3.9) and (3.11) into (3.8) yields that

$$\|z^+(t)\|_{\dot{F}_{p,q}^s} \leq \|z_0^+\|_{\dot{F}_{p,q}^s} + C \int_0^t (\|\nabla z^+\|_\infty + \|\nabla z^-\|_\infty) (\|z^-\|_{\dot{F}_{p,q}^s} + \|z^+\|_{\dot{F}_{p,q}^s}) d\tau. \quad (3.12)$$

Similar argument also leads to

$$\|z^-(t)\|_{\dot{F}_{p,q}^s} \leq \|z_0^-\|_{\dot{F}_{p,q}^s} + C \int_0^t (\|\nabla z^+\|_\infty + \|\nabla z^-\|_\infty) (\|z^-\|_{\dot{F}_{p,q}^s} + \|z^+\|_{\dot{F}_{p,q}^s}) d\tau. \quad (3.13)$$

In order to get the inhomogeneous version of (3.12) and (3.13), we have to estimate the  $L^p$  norm of  $(z^+, z^-)$ . Multiplying the first equation of (3.1) by  $|z^+|^{p-2}z^+$  and the second one by  $|z^-|^{p-2}z^-$ , integrating the resulting equations over  $\mathbf{R}^d$ , we obtain

$$\|z^+\|_p + \|z^-\|_p \leq \|z_0^+\|_p + \|z_0^-\|_p + C \int_0^t \|\nabla \pi(\tau)\|_p d\tau. \quad (3.14)$$

Using (3.10) and the  $L^p$ -boundedness of the Riesz transform, we get

$$\|\nabla \pi\|_p \leq C \|z^- \cdot \nabla z^+\|_p \leq C \|\nabla z^+\|_\infty \|z^-\|_p. \quad (3.15)$$

Summing up (3.12)-(3.15) yields that

$$\begin{aligned} \|z^+(t)\|_{F_{p,q}^s} + \|z^-(t)\|_{F_{p,q}^s} &\leq \|z_0^+\|_{F_{p,q}^s} + \|z_0^-\|_{F_{p,q}^s} \\ &+ C \int_0^t (\|\nabla z^+\|_\infty + \|\nabla z^-\|_\infty) (\|z^-\|_{F_{p,q}^s} + \|z^+\|_{F_{p,q}^s}) d\tau, \end{aligned} \quad (3.16)$$

which together with the Gronwall inequality gives

$$\|(z^+(t), z^-(t))\|_{F_{p,q}^s} \leq \|(z_0^+, z_0^-)\|_{F_{p,q}^s} \exp \left( C \int_0^t (\|\nabla z^+\|_\infty + \|\nabla z^-\|_\infty) d\tau \right). \quad (3.17)$$

**Step 2.** Approximate solutions and uniform estimates.

We construct the approximate solutions of (3.1). Define the sequence  $\{u^{(n)}, b^{(n)}\}_{\mathbf{N}_0 = \mathbf{N} \cup \{0\}}$  by solving the following systems:

$$\begin{cases} \partial_t u^{(n+1)} + u^{(n)} \cdot \nabla u^{(n+1)} - b^{(n)} \cdot \nabla b^{(n+1)} = -\nabla \tilde{\pi}_1^{(n+1)}, \\ \partial_t b^{(n+1)} + u^{(n)} \cdot \nabla b^{(n+1)} - b^{(n)} \cdot \nabla u^{(n+1)} = -\nabla \tilde{\pi}_2^{(n+1)}, \\ \nabla \cdot b^{(n+1)} = \nabla \cdot u^{(n+1)} = 0, \\ (u^{(n+1)}, b^{(n+1)})|_{t=0} = S_{n+2}(u_0, b_0). \end{cases} \quad (3.18)$$

We set  $(u^{(0)}, b^{(0)}) = (0, 0)$ , and

$$z^{+(n)} = u^{(n)} + b^{(n)}, \quad z^{-(n)} = u^{(n)} - b^{(n)}.$$

Then (3.18) can be reduced to

$$\begin{cases} \partial_t z^{+(n+1)} + (z^{-(n)} \cdot \nabla) z^{+(n+1)} = -\nabla \pi_1^{(n+1)}, \\ \partial_t z^{-(n+1)} + (z^{+(n)} \cdot \nabla) z^{-(n+1)} = -\nabla \pi_2^{(n+1)}, \\ \nabla \cdot z^{+(n+1)} = \nabla \cdot z^{-(n+1)} = 0, \quad \forall n \in \mathbf{N} \\ z^{+(n+1)}(0) = S_{n+2} z_0^+, \quad z^{-(n+1)}(0) = S_{n+2} z_0^-, \end{cases} \quad (3.19)$$

where  $(z^{+(0)}, z^{-(0)}) = (0, 0)$ . Similar to the proof of (3.16), we conclude that

$$\begin{aligned} &\|(z^{+(n+1)}(t), z^{-(n+1)}(t))\|_{F_{p,q}^s} \\ &\leq \|(z_0^+, z_0^-)\|_{F_{p,q}^s} + C \int_0^t \left( \|(\nabla z^{+(n)}, \nabla z^{-(n)})\|_\infty + \|(\nabla z^{+(n+1)}, \nabla z^{-(n+1)})\|_\infty \right) \\ &\quad \times \left( \|(\nabla z^{+(n)}, \nabla z^{-(n)})\|_{F_{p,q}^s} + \|(\nabla z^{+(n+1)}, \nabla z^{-(n+1)})\|_{F_{p,q}^s} \right) d\tau, \end{aligned} \quad (3.20)$$



where we used the fact that

$$\|(S_{n+2}z_0^+, S_{n+2}z_0^-)\|_{F_{p,q}^s} \leq \|(z_0^+, z_0^-)\|_{F_{p,q}^s}.$$

Note that  $F_{p,q}^{s-1} \hookrightarrow L^\infty$  for  $s-1 > \frac{d}{p}$ , (3.20) ensures that there exists  $T_0 = T_0(\|(z_0^+, z_0^-)\|_{F_{p,q}^s})$  such that for any  $n, t \in [0, T_0]$

$$\|(z^{+(n)}(t), z^{-(n)}(t))\|_{F_{p,q}^s} \leq 2\|(z_0^+, z_0^-)\|_{F_{p,q}^s}. \quad (3.21)$$

**Step 3. Existence.**

We will show that there exists a positive time  $T_1(\leq T_0)$  independent of  $n$  such that  $\{z^{+(n)}, z^{-(n)}\}$  is a Cauchy sequence in  $X_T^{s-1} \triangleq \mathcal{C}([0, T_1]; F_{p,q}^{s-1})$ . For this purpose, we set

$$\delta z^{+(n+1)} = z^{+(n+1)} - z^{+(n)}, \quad \delta z^{-(n+1)} = z^{-(n+1)} - z^{-(n)}, \quad \delta \pi_j^{(n+1)} = \pi_j^{(n+1)} - \pi_j^{(n)}, \quad j = 1, 2.$$

Using (3.19), it is easy to verify that the difference  $(\delta z^{+(n+1)}, \delta z^{-(n+1)}, \delta \pi^{(n)})$  satisfies

$$\begin{cases} \partial_t \delta z^{+(n+1)} + z^{-(n)} \cdot \nabla \delta z^{+(n+1)} = -\delta z^{-(n)} \cdot \nabla z^{+(n)} - \nabla \delta \pi_1^{(n+1)}, \\ \partial_t \delta z^{-(n+1)} + z^{+(n)} \cdot \nabla \delta z^{-(n+1)} = -\delta z^{+(n)} \cdot \nabla z^{-(n)} - \nabla \delta \pi_2^{(n+1)}, \\ (\delta z^{+(n+1)}, \delta z^{-(n+1)})|_{t=0} = \Delta_{n+1}(z_0^+, z_0^-). \end{cases} \quad (3.22)$$

Applying  $\Delta_k$  to the first equation of (3.22), we get

$$\begin{aligned} \partial_t \Delta_k \delta z^{+(n+1)} + z^{-(n)} \cdot \nabla \Delta_k \delta z^{+(n+1)} &= [z^{-(n)}, \Delta_k] \cdot \nabla \delta z^{+(n+1)} \\ &\quad - \Delta_k (\delta z^{-(n)} \cdot \nabla z^{+(n)}) - \nabla \Delta_k \delta \pi_1^{(n+1)}. \end{aligned} \quad (3.23)$$

Exactly as in the proof of (3.8), we get

$$\begin{aligned} \|\delta z^{+(n+1)}\|_{\dot{F}_{p,q}^{s-1}} &\leq C \|\Delta_{n+1} z_0^+\|_{\dot{F}_{p,q}^{s-1}} + \int_0^t \left\| 2^{k(s-1)} ([z^{-(n)}, \Delta_k] \cdot \nabla \delta z^{+(n+1)})(\alpha, \tau) \right\|_{\ell^q(\mathbf{Z})} \Big\|_p d\tau \\ &\quad + \int_0^t \|\delta z^{-(n)} \cdot \nabla z^{+(n)}(\tau)\|_{\dot{F}_{p,q}^{s-1}} d\tau + \int_0^t \|\nabla \delta \pi_1^{(n+1)}(\tau)\|_{\dot{F}_{p,q}^{s-1}} d\tau. \end{aligned} \quad (3.24)$$

Thanks to the Fourier support of  $\Delta_{n+1} z_0^+$ , we have

$$\|\Delta_{n+1} z_0^+\|_{\dot{F}_{p,q}^{s-1}} \leq C 2^{-(n+1)} \|z_0^+\|_{\dot{F}_{p,q}^s}. \quad (3.25)$$

Using Proposition 4.1 and the embedding  $F_{p,q}^{s-1} \hookrightarrow L^\infty$ , the second term on the right side of (3.24) is dominated by

$$\begin{aligned} &\|\nabla z^{-(n)}\|_\infty \|\delta z^{+(n+1)}\|_{\dot{F}_{p,q}^{s-1}} + \|\delta z^{+(n+1)}\|_\infty \|\nabla z^{-(n)}\|_{\dot{F}_{p,q}^{s-1}} \\ &\leq C \|z^{-(n)}\|_{F_{p,q}^s} \|\delta z^{+(n+1)}\|_{\dot{F}_{p,q}^{s-1}}. \end{aligned} \quad (3.26)$$

Thanks to Proposition 2.1, the third term on the right hand side of (3.24) is dominated by

$$\begin{aligned} &\|\delta z^{-(n)}\|_\infty \|\nabla z^{+(n)}\|_{\dot{F}_{p,q}^{s-1}} + \|\delta z^{-(n)}\|_{\dot{F}_{p,q}^{s-1}} \|\nabla z^{+(n)}\|_\infty \\ &\leq C \|\delta z^{-(n)}\|_{\dot{F}_{p,q}^{s-1}} \|z^{+(n)}\|_{F_{p,q}^s}. \end{aligned} \quad (3.27)$$

Taking the divergence on both sides of (3.22), we get

$$\delta\pi_1^{(n+1)} = \partial_j(-\Delta)^{-1}(\delta z_i^{-(n)}\partial_i z_j^{+(n)}) + \partial_i(-\Delta)^{-1}(\partial_j z_i^{-(n)}\delta z_j^{+(n+1)}).$$

Hence, we have

$$\partial_i\delta\pi_1^{(n+1)} = \mathcal{R}_l\mathcal{R}_j(\delta z_i^{-(n)}\partial_i z_j^{+(n)}) + \mathcal{R}_l\mathcal{R}_i(\partial_j z_i^{-(n)}\delta z_j^{+(n+1)}),$$

which together with Proposition 2.3 and the boundedness of the Riesz transform in the homogeneous Triebel-Lizorkin spaces gives

$$\begin{aligned} \|\nabla\delta\pi_1^{(n+1)}\|_{\dot{F}_{p,q}^{s-1}} &\lesssim \|\delta z_i^{-(n)}\partial_i z_j^{+(n)}\|_{\dot{F}_{p,q}^{s-1}} + \|\partial_j z_i^{-(n)}\delta z_j^{+(n+1)}\|_{\dot{F}_{p,q}^{s-1}} \\ &\lesssim \|\delta z^{-(n)}\|_\infty \|\nabla z^{+(n)}\|_{\dot{F}_{p,q}^{s-1}} + \|\delta z^{-(n)}\|_{\dot{F}_{p,q}^{s-1}} \|\nabla z^{+(n)}\|_\infty \\ &\quad + \|\nabla z^{-(n)}\|_\infty \|\delta z^{+(n+1)}\|_{\dot{F}_{p,q}^{s-1}} + \|\nabla z^{-(n)}\|_{\dot{F}_{p,q}^{s-1}} \|\delta z^{+(n+1)}\|_\infty \\ &\lesssim \|\delta z^{-(n)}\|_{F_{p,q}^{s-1}} \|z^{+(n)}\|_{F_{p,q}^s} + \|z^{-(n)}\|_{F_{p,q}^s} \|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}}. \end{aligned} \quad (3.28)$$

By summing up (3.24)-(3.28), we get

$$\begin{aligned} \|\delta z^{+(n+1)}\|_{\dot{F}_{p,q}^{s-1}} &\leq C2^{-(n+1)}\|z_0^+\|_{\dot{F}_{p,q}^s} + C \int_0^t \left( \|z^{-(n)}\|_{F_{p,q}^s} \|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}} \right. \\ &\quad \left. + \|\delta z^{-(n)}\|_{F_{p,q}^{s-1}} \|z^{+(n)}\|_{F_{p,q}^s} \right) dt. \end{aligned} \quad (3.29)$$

Now, we estimate the  $L^p$  norm of  $\delta z^{+(n+1)}$ . Multiplying  $|\delta z^{+(n+1)}|^{p-2}\delta z^{+(n+1)}$  on both sides of the first equation of (3.22), and integrating the resulting equations over  $\mathbf{R}^d$ , we obtain

$$\begin{aligned} \|\delta z^{+(n+1)}(t)\|_p &\leq \|\Delta_{n+1}z_0^+\|_p + \int_0^t \|\delta z^{-(n)} \cdot \nabla z^{+(n)}(\tau)\|_p d\tau + \int_0^t \|\nabla\delta\pi_1^{(n+1)}(\tau)\|_p d\tau \\ &\leq 2^{-(n+1)}\|z_0^+\|_{\dot{F}_{p,q}^s} + C \int_0^t \|\delta z^{-(n)}\|_p \|\nabla z^{+(n)}(\tau)\|_\infty d\tau \\ &\quad + C \int_0^t \|\nabla z^{-(n)}\|_\infty \|\delta z^{+(n+1)}\|_p d\tau, \end{aligned}$$

which together with (3.29) gives

$$\begin{aligned} \|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}} &\leq C2^{-(n+1)}\|z_0^+\|_{F_{p,q}^s} + C \int_0^t \left( \|z^{-(n)}\|_{F_{p,q}^s} \|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}} \right. \\ &\quad \left. + \|\delta z^{-(n)}\|_{F_{p,q}^{s-1}} \|z^{+(n)}\|_{F_{p,q}^s} \right) dt. \end{aligned} \quad (3.30)$$

Exactly as in the proof of (3.30), we also have

$$\begin{aligned} \|\delta z^{-(n+1)}\|_{F_{p,q}^{s-1}} &\leq C2^{-(n+1)}\|z_0^-\|_{F_{p,q}^s} + C \int_0^t \left( \|z^{+(n)}\|_{F_{p,q}^s} \|\delta z^{-(n+1)}\|_{F_{p,q}^{s-1}} \right. \\ &\quad \left. + \|\delta z^{+(n)}\|_{F_{p,q}^{s-1}} \|z^{-(n)}\|_{F_{p,q}^s} \right) dt. \end{aligned} \quad (3.31)$$

Adding up (3.30) and (3.31), we obtain

$$\begin{aligned}
& \|(\delta z^{+(n+1)}, \delta z^{-(n+1)})\|_{F_{p,q}^{s-1}} \\
& \lesssim 2^{-(n+1)}(\|z_0^+\|_{F_{p,q}^s} + \|z_0^-\|_{F_{p,q}^s}) \\
& \quad + T \sup_{t \in [0, T]} \|(z^{+(n)}, z^{-(n)})\|_{F_{p,q}^s} \|(\delta z^{+(n+1)}, \delta z^{-(n+1)})\|_{F_{p,q}^{s-1}} \\
& \quad + T \sup_{t \in [0, T]} \|(z^{+(n)}, z^{-(n)})\|_{F_{p,q}^s} \|(\delta z^{+(n)}, \delta z^{-(n)})\|_{F_{p,q}^{s-1}},
\end{aligned}$$

which together with (3.21) yields that

$$\begin{aligned}
\|(\delta z^{+(n+1)}, \delta z^{-(n+1)})\|_{X_T^{s-1}} & \leq C_1 2^{-(n+1)} + C_1 T \|(\delta z^{+(n+1)}, \delta z^{-(n+1)})\|_{X_T^{s-1}} \\
& \quad + C_1 T \|(\delta z^{+(n)}, \delta z^{-(n)})\|_{X_T^{s-1}},
\end{aligned} \tag{3.32}$$

where  $C_1 = C_1(\|z_0^+, z_0^-\|_{F_{p,q}^s})$ . Thus, if  $C_1 T \leq \frac{1}{4}$ , then

$$\|(\delta z^{+(n+1)}, \delta z^{-(n+1)})\|_{X_T^{s-1}} \leq C_1 2^{-n} + 2C_1 T \|(\delta z^{+(n)}, \delta z^{-(n)})\|_{X_T^{s-1}}.$$

This implies that

$$\|(\delta z^{+(n+1)}, \delta z^{-(n+1)})\|_{X_T^{s-1}} \leq 2C_1 2^{-(n+1)}.$$

Thus,  $\{z^{+(n)}, z^{-(n)}\}_{n \in \mathbf{N}_0}$  is a Cauchy sequence in  $X_{T_1}^{s-1}$ . By the standard argument, for  $T_1 \leq \min\{T_0, \frac{1}{4C_1}\}$ , the limit  $(z^+, z^-) \in X_{T_1}^s$  solves the equation (3.1) with the initial data  $(z_0^+, z_0^-)$ . Moreover,  $(z^+, z^-)$  satisfies

$$\|(z^+, z^-)(t)\|_{L_{T_1}^\infty(F_{p,q}^s)} \leq C\|(z_0^+, z_0^-)\|_{F_{p,q}^s},$$

which implies  $(u, b)$  is a solution of (1.1) with the initial data  $(u_0, b_0) \in F_{p,q}^s$ , and

$$\|(u, b)(t)\|_{L_{T_1}^\infty(F_{p,q}^s)} \leq C\|(u_0, b_0)\|_{F_{p,q}^s}.$$

**The proof of the uniqueness.** Consider  $(z^{+'}, z^{-'}) \in C_{T_1}(F_{p,q}^s)$  is another solution to (3.1) with the same initial data. Let  $\delta z^+ = z^+ - z^{+'}$  and  $\delta z^- = z^- - z^{-'}$ . Then  $(\delta z^+, \delta z^-)$  satisfies the following equations

$$\begin{cases} \partial_t \delta z^+ + (z^- \cdot \nabla) \delta z^+ = -(\delta z^- \cdot \nabla) z^+ - \nabla(\pi - \pi'), \\ \partial_t \delta z^- + (z^+ \cdot \nabla) \delta z^- = -(\delta z^+ \cdot \nabla) z^- - \nabla(\pi - \pi'), \\ \nabla \cdot \delta z^+ = \nabla \cdot \delta z^- = 0. \end{cases}$$

In the same way as deriving in (3.32), we obtain

$$\|(\delta z^+, \delta z^-)\|_{X_T^{s-1}} \leq C_2 T \|(\delta z^+, \delta z^-)\|_{X_T^{s-1}}$$

for sufficiently small  $T$ . This implies that  $(\delta z^+, \delta z^-) \equiv 0$ , i.e.,  $(z^+, z^-) \equiv (z^{+'}, z^{-'})$ .

**Blow-up Criterion.** By means of Proposition 1.1 in [7] and

$$\|(\nabla z^+, \nabla z^-)\|_{\dot{F}_{\infty,\infty}^0} \lesssim \|(\nabla \times z^+, \nabla \times z^-)\|_{\dot{F}_{\infty,\infty}^0},$$

we have

$$\|(\nabla z^+, \nabla z^-)\|_{\infty} \lesssim \left(1 + \|(\nabla z^+, \nabla z^-)\|_{\dot{F}_{\infty,\infty}^0} \left(\log(1 + \|(\nabla \times z^+, \nabla \times z^-)\|_{F_{p,q}^{s-1}}) + 1\right)\right)$$

Plugging the above estimates into (3.16) then by Gronwall's lemma yields that

$$\|(z^+, z^-)\|_{F_{p,q}^s} \leq \|(z_0^+, z_0^-)\|_{F_{p,q}^s} \exp \left[ C \exp \left[ C \int_0^t (1 + \|(\nabla \times z^+, \nabla \times z^-)\|_{\dot{F}_{\infty,\infty}^0} d\tau) \right] \right]$$

which implies the blow-up criterion. This finishes the proof of the Theorem 1.1.

## 4 Appendix

Let us recall the para-differential calculus which enables us to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see [3]). The para-product between  $u$  and  $v$  is defined by

$$T_u v \triangleq \sum_{j \in \mathbf{Z}} S_{j-1} u \Delta_j v.$$

We then have the following formal decomposition:

$$uv = T_u v + T_v u + R(u, v), \quad (4.33)$$

with

$$R(u, v) = \sum_{j \in \mathbf{Z}} \Delta_j u \tilde{\Delta}_j v \quad \text{and} \quad \tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}.$$

The decomposition (4.33) is called the Bony's para-product decomposition.

**Proposition 4.1** *Let  $(p, q) \in (1, \infty) \times (1, \infty]$ , or  $p = q = \infty$ , and  $f$  be a solenoidal vector field. Then for  $s > 0$*

$$\left\| \left\| 2^{ks} ([f, \Delta_k] \cdot \nabla g) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \lesssim (\|\nabla f\|_{\infty} \|g\|_{\dot{F}_{p,q}^s} + \|\nabla g\|_{\infty} \|f\|_{\dot{F}_{p,q}^s}). \quad (4.34)$$

or for  $s > -1$

$$\left\| \left\| 2^{ks} ([f, \Delta_k] \cdot \nabla g) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \lesssim (\|\nabla f\|_{\infty} \|g\|_{\dot{F}_{p,q}^s} + \|g\|_{\infty} \|\nabla f\|_{\dot{F}_{p,q}^s}). \quad (4.35)$$

*Proof.* By the Einstein convention on the summation over repeated indices  $i \in [1, d]$ , and the Bony's paraproduct decomposition we decompose

$$\begin{aligned} [f, \Delta_k] \cdot \nabla g &= [f_i, \Delta_k] \partial_i g = [T_{f_i}, \Delta_k] \partial_i g + T'_{\Delta_k \partial_i g} f_i - \Delta_k (T_{\partial_i g} f_i) - \Delta_k (R(f_i, \partial_i g)) \\ &\triangleq I + II + III + IV, \end{aligned}$$

where  $T'_u v$  stands for  $T_u v + R(u, v)$ . Thank to the support condition (2.1), we rewrite

$$\begin{aligned} |I| &= \left| \sum_{k' \sim k} [S_{k'-1} f_i, \Delta_k] \partial_i \Delta_{k'} g \right| \\ &= \left| \sum_{k' \sim k} \int_{\mathbf{R}^d} \left( S_{k'-1} f_i(x) - S_{k'-1} f_i(y) \right) 2^{kd} h(2^k(x-y)) \partial_i \Delta_{k'} g(y) dy \right|, \end{aligned} \quad (4.36)$$

where  $k' \sim k$  stands for  $|k' - k| \leq 4$ . Integrate by part and use  $\operatorname{div} f = 0$ , the integrand in (4.36) is

$$\left( S_{k'-1} f_i(x) - S_{k'-1} f_i(y) \right) 2^{k(d+1)} (\partial_i h)(2^k(x-y)) \Delta_{k'} g(y)$$

which was dominated by

$$\|\nabla S_{k'-1} f\|_\infty 2^k |x-y| 2^{kd} |\nabla h(2^k(x-y))| |\Delta_{k'} g(y)|. \quad (4.37)$$

Recall  $h(x) \in \mathcal{S}(\mathbf{R}^d)$ , it is easy to see that  $|x \nabla h(x)|$  satisfies Lemma 2.5, so (4.37) is less than

$$C \|\nabla S_{k'-1} f\|_\infty M(|\Delta_{k'} g(\cdot)|)(x). \quad (4.38)$$

Multiplying  $2^{ks}$  on both sides of (4.36), taking  $\ell^q(\mathbf{Z})$  norm then taking  $L^p$  norm and putting (4.38) into the resulting inequality, we have

$$\begin{aligned} \left\| \|2^{ks} |I(x)|\|_{\ell^q(\mathbf{Z})} \right\|_p &\lesssim \|\nabla S_{k'-1} f\|_\infty \left\| \left\| \sum_{k' \sim k} 2^{(k-k')s} M(2^{k's} |\Delta_{k'} g(\cdot)|)(x) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\ &\lesssim \|\nabla f\|_\infty \left\| \left\| M(2^{k's} |\Delta_{k'} g(\cdot)|)(x) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\ &\lesssim \|\nabla f\|_\infty \left\| \|2^{ks} |\Delta_k g(x)|\|_{\ell^q(\mathbf{Z})} \right\|_p \lesssim \|\nabla f\|_\infty \|g\|_{\dot{F}_{p,q}^s}, \end{aligned} \quad (4.39)$$

where we used Lemma 2.4 in the third inequality. Let us turn to the term  $II$ , thanks to the definition of  $II$ ,

$$|II| = \left| \sum_{k' \geq k-2} S_{k'+2} \partial_i \Delta_k g \Delta_{k'} f_i(x) \right| \leq \sum_{k' \geq k-2} \|\nabla \Delta_k g\|_\infty |\Delta_{k'} f(x)|. \quad (4.40)$$

Then thanks to the convolution inequality for series, we get for  $s > 0$ ,

$$\begin{aligned} \left\| \|2^{ks} |II(x)|\|_{\ell^q(\mathbf{Z})} \right\|_p &\lesssim \|\nabla \Delta_k g\|_\infty \left\| \left\| \sum_{k' \geq k-2} 2^{(k-k')s} 2^{k's} |\Delta_{k'} f(x)| \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\ &\lesssim \|\nabla \Delta_k g\|_\infty \left\| \|2^{-ks} \chi_{\{k \geq -2\}}\|_{\ell^1(\mathbf{Z})} \|2^{k's} |\Delta_{k'} f(x)|\|_{\ell^q(\mathbf{Z})} \right\|_p \\ &\lesssim \|\nabla g\|_\infty \left\| \|2^{ks} |\Delta_k f(x)|\|_{\ell^q(\mathbf{Z})} \right\|_p \lesssim \|\nabla g\|_\infty \|f\|_{\dot{F}_{p,q}^s}. \end{aligned} \quad (4.41)$$

For the term  $III$ ,

$$\begin{aligned} |III| &= \left| \sum_{k' \sim k} \Delta_k (S_{k'-1} \partial_i g \Delta_{k'} f_i) \right| \lesssim \sum_{k' \sim k} |M(S_{k'-1} \partial_i g \Delta_{k'} f_i)(x)| \\ &\lesssim \sum_{k' \sim k} |M(|\Delta_{k'} f|)(x)| \|S_{k'-1} \nabla g\|_\infty. \end{aligned} \quad (4.42)$$

Using (4.42) and in the same way as leading to (4.39) yields

$$\begin{aligned}
\| \|2^{ks} |III(x)| \|_{\ell^q(\mathbf{Z})} \|_p &\lesssim \| \nabla S_{k'-1} g \|_\infty \left\| \left\| \sum_{k' \sim k} 2^{(k-k')s} M(2^{k's} |\Delta_{k'} f|)(x) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\
&\lesssim \| \nabla g \|_\infty \left\| \left\| M(2^{k's} |\Delta_{k'} f|)(x) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\
&\lesssim \| \nabla g \|_\infty \| \|2^{ks} |\Delta_k f(x)| \|_{\ell^q(\mathbf{Z})} \|_p \lesssim \| \nabla g \|_\infty \| f \|_{\dot{F}_{p,q}^s}. \tag{4.43}
\end{aligned}$$

In view of  $\operatorname{div} f = 0$  and integrating by part, we have

$$\begin{aligned}
|IV| &= \left| \sum_{k' \geq k-3} \Delta_k (\Delta_{k'} f_i \partial_i \tilde{\Delta}_{k'} g) \right| = \left| \sum_{k' \geq k-3} \int_{\mathbf{R}^d} 2^{kd} h(2^k(x-y)) \Delta_{k'} f_i(y) \partial_i \tilde{\Delta}_{k'} g(y) dy \right| \\
&= \left| \sum_{k' \geq k-3} \int_{\mathbf{R}^d} 2^{kd+k} (\partial_i h)(2^k(x-y)) \Delta_{k'} f_i(y) \tilde{\Delta}_{k'} g(y) dy \right| \\
&\lesssim \sum_{k' \geq k-3} 2^k M(\Delta_{k'} f \tilde{\Delta}_{k'} g)(x) \lesssim \sum_{k' \geq k-3} 2^k M(\tilde{\Delta}_{k'} g)(x) \| \Delta_{k'} f \|_\infty. \tag{4.44}
\end{aligned}$$

The convolution inequality for series and Lemma 2.4 allow us to give that for  $s+1 > 0$ ,

$$\begin{aligned}
\| \|2^{ks} |IV(x)| \|_{\ell^q(\mathbf{Z})} \|_p &\lesssim \| \nabla \Delta_{k'} f \|_\infty \left\| \left\| \sum_{k' \geq k-3} 2^{(k-k')(s+1)} M(2^{k's} \tilde{\Delta}_{k'} g)(x) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\
&\lesssim \| \nabla f \|_\infty \left\| \left\| M(2^{k's} \tilde{\Delta}_{k'} g)(x) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\
&\lesssim \| \nabla f \|_\infty \| \|2^{ks} |\tilde{\Delta}_k g(x)| \|_{\ell^q(\mathbf{Z})} \|_p \lesssim \| \nabla f \|_\infty \| g \|_{\dot{F}_{p,q}^s}. \tag{4.45}
\end{aligned}$$

Summing up (4.39), (4.41), (4.43) and (4.45), we get the desired inequality (4.34).

In order to prove the inequality (4.35), we only indicate how to get the bound on  $II$  and  $III$  since  $I$  and  $IV$  can be treated as above. We estimate the term  $II$  as

$$|II| = \left| \sum_{k' \geq k-2} S_{k'+2} \partial_i \Delta_k g \Delta_{k'} f_i(x) \right| \leq \sum_{k' \geq k-2} 2^k \| \Delta_k g \|_\infty | \Delta_{k'} f(x) |.$$

Then thanks to the convolution inequality for series, we get for  $s+1 > 0$ ,

$$\begin{aligned}
\| \|2^{ks} |II(x)| \|_{\ell^q(\mathbf{Z})} \|_p &\lesssim \| \Delta_k g \|_\infty \left\| \left\| \sum_{k' \geq k-2} 2^{(k-k')(s+1)} 2^{k'(s+1)} | \Delta_{k'} f(x) | \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\
&\lesssim \| g \|_\infty \left\| \left\| 2^{-k(s+1)} \chi_{\{k \geq -2\}} \right\|_{\ell^1(\mathbf{Z})} \left\| 2^{k'(s+1)} | \Delta_{k'} f(x) | \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\
&\lesssim \| g \|_\infty \| f \|_{\dot{F}_{p,q}^{s+1}}.
\end{aligned}$$

Let's turn to the term  $III$ ,

$$|III| \lesssim \sum_{k' \sim k} |M(|\Delta_{k'} f|)(x)| 2^{k'} \| S_{k'-1} g \|_\infty.$$

Arguing similarly as in deriving (4.43) yields that

$$\begin{aligned}
\| \|2^{ks} |III(x)| \|_{\ell^q(\mathbf{Z})} \|_p &\lesssim \| S_{k'-1} g \|_\infty \left\| \left\| \sum_{k' \sim k} 2^{(k-k')s} M(2^{k'(s+1)} |\Delta_{k'} f|)(x) \right\|_{\ell^q(\mathbf{Z})} \right\|_p \\
&\lesssim \| g \|_\infty \| f \|_{\dot{F}_{p,q}^{s+1}}.
\end{aligned}$$

Thus the desired inequality (4.35) is obtained.  $\square$

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